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Mean Square Response to Band-Limited White Noise Excitation

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Introduction

THE stationary response of a stable time-invariant linear multidegree-of-freedom (MDF) system to white noise and second-order filtered white noise excitation was studied by the authors[1,2] via complex modal anlaysis. However, an important theoretical (as well as practical) case, i.e, that for band-limited white noise excitation, has not been addressed. An exact solution for the mean square response of a single degree-of-freedom system to band-limited white noise was given by Crandall and Mark[3] about 20 years ago. Since then its counterpart for MDF systems has not appeared. However, an exact solution for MDF systems based on Ref. 1 is given here.

The System Response

The system considered is time-invariant linear MDF, either classically damped or not and symmetrical or not. Its differential equation can be described as follows:

$$m\ddot{y}+c\dot{y}+ky=f(t) \tag{1}$$

where y is the system response; m, c, and k the mass, damping, and stiffness matrices, respectively; and $f(t)$ the band-limited white noise excitation with zero mean and the following autospectral matrix:

$$S_f(\omega)=D, \quad \text{when } \omega_1<|\omega|<\omega_2$$
$$=0, \quad \text{elsewhere} \tag{2}$$

where D is a real symmetrical non-negative constant matrix.

The system response may be found in a somewhat indirect way. First, we find the system response x to white noise excitation $w(t)$ with zero mean and the autospectral matrix

$$S_w(\omega)=D$$

Its correlation function matrix is

$$R_w(\tau)=2\pi D\delta(\tau)$$

In this case, the differential equation is

$$m\ddot{x}+c\dot{x}+kx=w(t) \tag{3}$$

When the damping is below critical, all of the system eigenvalues appear as complex conjugates with negative real part

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and the eigenvalue matrix may then be written as

$$P=\operatorname{diag}[p_i,\bar{p}_j],\qquad i,j=1,\ldots,n$$

The corresponding right and left modal matrices are as follows:

$$u=[u_1\ldots u_n\bar{u}_1\ldots\bar{u}_n]\quad\text{(right)}$$

$$v=[v_1\ldots v_n\bar{v}_1\ldots\bar{v}_n]\quad\text{(left)}$$

By introducing state variables and the complex modal transform

$$x=uz$$

Eq. (3) can be reduced to a system of first-order equations with the complex modal response z,

$$\dot{z}-Pz=F(t)\tag{4}$$

where

$$F(t)=M^{-1}v^Tw(t)$$

$$M=\operatorname{diag}[m_i,\bar{m}_j],\qquad i,j=1,\ldots,n$$

$$m_i=v_i^T[2p_im+c]u_i,\quad i=1,\ldots,n$$

Solving Eq. (4), we have the correlation function matrix of z as follows:

$$R_z(\tau)=[r_{is}]e^{\bar{P}\tau},\quad \tau>0\tag{5}$$

$$r_{is}=-2\pi g_{is}/(p_i+\bar{p}_s),\quad i,s=1,\ldots,2n$$

where

$$[g_{is}]=M^{-1}v^TD\bar{v}\bar{M}^{-T}$$

Returning to the system response, x, we have

$$R_x(\tau)=uR_z(\tau)\bar{u}^T$$

For our present purpose, we would rather convert this result into another form by dividing $[r_{is}]$ into four equidimensional submatrices:

$$[r_{is}]=\begin{bmatrix} r^a & r^b\\ r^c & r^d\end{bmatrix}$$

Then we have

$$R_x(\tau)=\sum_{i=1}^{n}(C_ie^{p_i\tau}+\bar{C}_ie^{\bar{p}_i\tau}),\quad \tau>0$$

where

$$\bar{C}_i=\sum_{k=1}^{n}[r_{ki}^au_k\bar{u}_i^T+r_{ki}^c\bar{u}_k\bar{u}_i^T]$$

Note that D is real symmetrical, while $[g_{is}]$ and $[r_{is}]$ are Hermitian. Also, we have

$$r^a=\bar{r}^d,\qquad r^b=\bar{r}^c$$

The elements on the diagonal of $R_x(\tau)$, i.e., the correlation functions of the individual responses, have the following form:

$$R_{xk}(\tau)=\sum_{i=1}^{n}(c_{ki}e^{p_i|\tau|}+\bar{c}_{ki}e^{\bar{p}_i|\tau|}),\quad k=1,\ldots,n\tag{6}$$

where

$$\bar{c}_{ki}=\sum_{j=1}^{n}(r_{ji}^au_{kj}\bar{u}_{ki}+r_{ji}^c\bar{u}_{kj}\bar{u}_{ki})$$

In other words, the response x_k may be regarded as a linear combination of n independent second-order filtered white noise responses, each of which has a correlation function,

$$R_{ki}(\tau)=c_{ki}e^{p_i|\tau|}+\bar{c}_{ki}e^{\bar{p}_i|\tau|}$$

Since Eq. (6) is simply a linear operation, its Fourier transform $S_{xk}(\omega)$ can be found term by term.

It is not difficult to find the Fourier transform for the ith term, $R_{ki}(\tau)$. By noting the real and imaginary parts

$$p_i=-a_i+jb_i,\qquad a_i>0$$

$$c_{ki}=A_i+jB_i$$

$R_{ki}(\tau)$ can be written as

$$R_{ki}(\tau)=2A_ie^{-a_i|\tau|}\cos(b_i\tau)-2B_ie^{-a_i|\tau|}\sin(b_i|\tau|)\tag{7}$$

Its Fourier transform is

$$\begin{aligned}S_{ki}(\omega)=&\frac{1}{\pi}\left[\frac{a_i}{a_i^2+(\omega-b_i)^2}+\frac{a_i}{a_i^2+(\omega+b_i)^2}\right]A_i\\&+\frac{1}{\pi}\left[\frac{\omega-b_i}{a_i^2+(\omega-b_i)^2}+\frac{\omega+b_i}{a_i^2+(\omega+b_i)^2}\right]B_i\end{aligned}\tag{8}$$

Finally, we obtain

$$S_{xk}(\omega)=\sum_{i=1}^{n}S_{ki}(\omega)$$

It is easy to verify that

$$\int_{-\infty}^{\infty}S_{ki}(\omega)\mathrm{d}\omega=R_{ki}(0)$$

Now return to the band-limited white noise excitation case. In view of Eq. (2), the power spectral density for response y_k to band-limited white noise can be written as

$$S_{yk}(\omega)=\sum_{i=1}^{n}S_{ki}^y(\omega)\tag{9}$$

where

$$\begin{aligned}S_{ki}^y(\omega)&=S_{ki}(\omega),\quad\text{when }\omega_1>|\omega|<\omega_2\\&=0,\qquad\text{elsewhere}\end{aligned}$$

The mean square response, $\langle y_k^2\rangle$, can be obtained through the following integration:

$$\langle y_k^2\rangle=\int_{-\infty}^{\infty}S_{yk}(\omega)\mathrm{d}\omega\tag{10}$$

By substituting Eq. (9) into Eq. (10), the integration can be done term by term. The ith term for $\langle y_k^2\rangle$ can be obtained as

$$\langle y_{ki}^2\rangle=\frac{2}{\pi}[A_i(X_{2i}-X_{1i})-B_i(Z_{2i}-Z_{1i})]$$

Table 1 Numerical results

ω_2	$\langle x_1^2 \rangle$	$\langle x_2^2 \rangle$	ω_2	$\langle x_1^2 \rangle$	$\langle x_2^2 \rangle$
5	0.000042	0.000173	30	0.001777	0.006593
10	0.000095	0.000386	35	0.001837	0.007116
12	0.000124	0.000500	40	0.001883	0.007607
15	0.000187	0.000744	50	0.001906	0.007831
20	0.000557	0.002094	100	0.001916	0.007910
22	0.001177	0.004305	200	0.001916	0.007916
25	0.001652	0.006025	∞	0.001916	0.007917

where

$$X_{ji} = \tan^{-1}\frac{\omega_j + b_i}{a_i} + \tan^{-1}\frac{\omega_j - b_i}{a_i} \qquad j=1,2$$

$$Z_{ji} = \frac{1}{2}\ln\frac{a_i^2 + (\omega_j + b_i)^2}{a_i^2 + (\omega_j - b_i)^2} \qquad j=1,2$$

Example

Consider a two-degree-of-freedom classically damped linear system subjected to band-limited white noise excitation. The governing differential equation may be expressed as Eq. (1), with

$$m = \begin{bmatrix} 100 & 0 \\ 0 & 15 \end{bmatrix}, \quad c = \begin{bmatrix} 500 & -100 \\ -100 & 100 \end{bmatrix}, \quad k = 150c$$

The excitation $f(t)$ may be characterized by Eq. (2), with $2\pi D = 100$ m, $\omega_1 = 0$, and ω_2 taking different values of 5-200.

The four eigenvalues of the system are

$$p_1 = \bar{p}_3 = -a_1 + jb_1 = -1.5601 + j21.5777$$

$$p_2 = \bar{p}_4 = -a_2 + jb_2 = -4.2732 + j35.5487$$

The correlation functions of the system response due to white noise excitation can be found as

$$\langle x_s(t)x_s(t+\tau)\rangle = \sum_{i=1}^{2} Re[(A_{si} + jB_{si})e^{p_i|\tau|}], \quad s=1,2$$

where

$$A_{si} = \frac{a_{si}^2}{a_i(a_i^2 + b_i^2)}, \quad B_{si} = -A_{si}\left(\frac{a_i}{b_i}\right)$$

with

$$a_{11} = 0.8084, \quad a_{12} = 0.5886$$

$$a_{21} = 1.5179, \quad a_{22} = -2.0874$$

Then the mean square response of the system due to band-limited white noise excitation can be formulated as

$$\langle x_s^2 \rangle = \frac{2}{\pi}\sum_{i=1}^{2} A_{si}X_{2i} - B_{si}Z_{2i}$$

The numerical results are listed in Table 1. Let $y = \langle x_1^2 \rangle + \langle x_2^2 \rangle$ be a level evaluation of the system response and let y^0, the response level due to white noise excitation, be one unit. From Table 1, we can see that 97.5% of y^0 lies in the low-pass band below $\omega = 40$. Roughly speaking, only about 10% of y^0 lies in the low-pass band below $\omega = 15$ and about 20% in the pass-band between $\omega = 25$ and 40. In this example, the random vibrations around the first natural frequency, ω_1 $(=21.634)$, are dominant. It is seen that two-thirds or more of y^0 is concentrated in a narrow band between $\omega = 15$ and 25, the bandwidth of which is only 1.6 Hz.

Conclusions

1) The mean square response to band-limited white noise for MDF systems has been given in analytical form via complex modal analysis.

2) The method is applicable to any time-invariant linear system, whether classically damped or not or symmetrical or not.

3) Band-limited white noise is a convenient mathematical model and also useful to obtain by superposition an approximation for other noise spectra. Band-limited white noise can be readily physically realized and is commonly used for simulation experiments. The exact mean square response to band-limited white noise excitation can be used as a standard to compare with experimental results or to estimate the error involved in the ideal white noise excitation approximation.

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